

Recurrences for certain sequences of binomial sums in terms of (generalized) Fibonacci and Lucas polynomials

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Abstract.

We give a simplified presentation of some results about recurrences of certain sequences of binomial sums in terms of (generalized) Fibonacci and Lucas polynomials.

1. Introduction

Motivated by the remarkable identities

$$F_n = \sum_{j \in \mathbb{Z}} (-1)^j \binom{n}{\lfloor \frac{n+5j+2}{2} \rfloor} \quad (1)$$

and

$$F_{n+1} = \sum_{j \in \mathbb{Z}} (-1)^j \binom{n}{\lfloor \frac{n+5j}{2} \rfloor} \quad (2)$$

for the Fibonacci numbers I obtained in [4] - [9] recurrences for binomial sums of the form

$$A_n(k, m, \ell, z) = \sum_{h \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+mh+\ell}{k} \rfloor} z^h \quad (3)$$

in terms of (generalized) Fibonacci and Lucas polynomials. Using some results of John P. D'Angelo [2], Eduardo H. M. Brietzke [3] and Dusty E. Grundmeier [10] we give a simplified derivation of these recurrences. As a byproduct we obtain recurrences of subsequences of generalized Fibonacci polynomials. To make the paper accessible to a wider readership, we also provide proofs of some known results that are not part of common knowledge.

2. The case $k=2$

The recurrences of the sequences $(A_n(2, m, \ell, z))_{n \geq 0}$ can be expressed using well-known properties of Fibonacci and Lucas polynomials.

The Fibonacci polynomials

$$F_n(x, s) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-j}{j} s^j x^{n-1-2j} \quad (4)$$

satisfy $F_n(x, s) = xF_{n-1}(x, s) + sF_{n-2}(x, s)$ with initial values $F_0(x, s) = 0$ and $F_1(x, s) = 1$.

The Lucas polynomials

$$L_n(x, s) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} \frac{n}{n-j} s^j x^{n-2j} \quad (5)$$

satisfy $L_n(x, s) = xL_{n-1}(x, s) + sL_{n-2}(x, s)$ with initial values $L_0(x, s) = 2$ and $L_1(x, s) = x$.

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Binet's formulae give

$$F_n(x, s) = \frac{\alpha^n(x, s) - \beta^n(x, s)}{\alpha - \beta} \quad (6)$$

and

$$L_n(x, s) = \alpha^n(x, s) + \beta^n(x, s) \quad (7)$$

with

$$\alpha(x, s) = \frac{x + \sqrt{x^2 + 4s}}{2}, \quad \beta(x, s) = \frac{x - \sqrt{x^2 + 4s}}{2}. \quad (8)$$

Since

$$\alpha(x + y, -xy) = \frac{x + y + \sqrt{(x + y)^2 - 4xy}}{2} = x, \quad \beta(x + y, -xy) = \frac{x + y - \sqrt{(x + y)^2 - 4xy}}{2} = y,$$

we get the well-known formulas

$$L_n(x + y, -xy) = x^n + y^n \quad (9)$$

and

$$F_n(x + y, -xy) = \frac{x^n - y^n}{x - y}. \quad (10)$$

Let Δ denote the difference operator $\Delta f(x) = f(x+1) - f(x)$ and E the translation operator

$Ef(x) = f(x+1)$ on the vector space of polynomials which satisfy $E^i \binom{x}{r} = \binom{x+i}{r}$ and

$$\Delta^j \binom{x}{r} = \binom{x}{r-j}.$$

By (9) we get $L_m(x+1, -x) = x^m + 1$ which for $x = \Delta$ gives

$$L_m(E, -\Delta) = \Delta^m + I, \quad (11)$$

where I denotes the identity operator.

Applying (11) to the polynomial $\binom{x}{r}$ gives

$$L_m(E, -\Delta) \binom{x}{r} = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-j}{j} \frac{m}{m-j} (-1)^j \binom{x+m-2j}{r-j} = \binom{x}{r-m} + \binom{x}{r}.$$

Choosing $x = n$, $r = \lfloor \frac{n+mh+\ell}{2} \rfloor$, multiplying both sides with z^h and summing over all $h \in \mathbb{Z}$

we get

$$\sum_{h \in \mathbb{Z}} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-j}{j} \frac{m}{m-j} (-1)^j \left\lfloor \frac{n+m-2j}{n-2j+m+m(h-1)+\ell} \right\rfloor z^h = \sum_{h \in \mathbb{Z}} \left\lfloor \frac{n}{n+m(h-2)+\ell} \right\rfloor z^h + \sum_{h \in \mathbb{Z}} \left\lfloor \frac{n}{n+mh+\ell} \right\rfloor z^h.$$

Changing the order of summation gives

$$z \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-j}{j} \frac{m}{m-j} (-1)^j A_{n+m-2j}(2, m, \ell, z) = (1+z^2) A_n(2, m, \ell, z).$$

This can be written as

$$\left(zL_m(N, -1) - (1+z^2) \right) A_n(2, m, \ell, z) = 0, \quad (12)$$

if N denotes the operator defined by $N^i A_n = A_{n+i}$.

If $p(N)f(n) = 0$ for a polynomial $p(t)$ we call $p(t)$ the *characteristic polynomial of the sequence* ($f(n)$). Note that $p(t)$ is unique up to a multiplicative constant.

Theorem 1

The characteristic polynomial of the sequence $A_n(2, m, \ell, z)$ is

$$p(t) = (1 + z^2) - zL_m(t, -1). \tag{13}$$

Since $L_5(x, -1) = x^5 - 5x^3 + 5x$ each sequence $A_n(2, 5, \ell, -1)$ satisfies the recurrence

$$(N^5 - 5N^3 + 5N + 2)A_n(2, 5, \ell, -1) = 0. \text{ Observing that}$$

$$N^5 - 5N^3 + 5N + 2 = (N + 2)(N^2 - N - 1)^2 \text{ and } (N^2 - N - 1)F_{n+i} = 0 \text{ for each } i \text{ identities (1)}$$

and (2) follow from the fact that $A_n(2, 5, 2, -1) = F_n$ and $A_n(2, 5, 0, -1) = F_{n+1}$ for $0 \leq n \leq 4$.

Remark

The identities (1) and (2) have been found by George Andrews [1]. The proof of the Rogers-Ramanujan identities by Issai Schur [12] contains a q - analog of these identities.

For $z = \pm 1$ simpler recurrences will be obtained in paragraph 4 with other methods.

The numbers $A_n(2, m, 0, -1)$ can be interpreted (cf. e.g. [9]) among other things as the number

of the set of all lattice paths in \mathbb{Z}^2 which start at the origin, consist of $\lfloor \frac{n}{2} \rfloor$ north-east steps

$U = (1, 1)$ and $\lfloor \frac{n+1}{2} \rfloor$ south-east steps $D = (1, -1)$ and are contained in the strip

$-\lfloor \frac{m-1}{2} \rfloor \leq y \leq \lfloor \frac{m-2}{2} \rfloor$. For small m these sequences occur in many contexts (cf. OEIS[11]):

m	OEIS	first terms
3	A000012	1, 1, 1, 1, 1, ...
4	A016116	1, 1, 2, 2, 4, 4, 8, 8, ...
5	A000045	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...
6	A182522	1, 1, 2, 3, 6, 9, 18, 27, 54, 81, 162, ...
7	A028495	1, 1, 2, 3, 6, 10, 19, 33, 61, 108, 197, ...
8	A030436	1, 1, 2, 3, 6, 10, 20, 34, 68, 116, 232, 396, ...
9	A061551	1, 1, 2, 3, 6, 10, 20, 35, 69, 124, 241, 440, 846, ...

For $z = 1$ and $\ell = 0$ we get

m	OEIS	first terms
2	A000079	1, 2, 4, 8, 16, 32, ...
3	A001045	1, 1, 3, 5, 11, 21, 43, 85, ...
4	A011782	1, 1, 2, 4, 8, 16, 32, 64, 128, ...
5	A099163	1, 1, 2, 3, 7, 12, 27, 49, 106, 199, ...
6	A005578	1, 1, 2, 3, 6, 11, 22, 43, 86, 171, 342, 683, ...

$A_{2n}(2, 2m, 0, 1) = \sum_{h \in \mathbb{Z}} \binom{2n}{n+hm}$ can be interpreted as the number of closed walks of length $2n$

on a vertex of the cyclic graph on $2m$ nodes or equivalently as the number of lattice paths with steps U and D from $(0, 0)$ to $(2n, 2hm)$ for some $h \in \mathbb{Z}$.

$A_{2n+1}(2, 2m, 0, 1) = \sum_{h \in \mathbb{Z}} \binom{2n+1}{n+hm}$ is the number of walks of length $2n+1$ between two adjacent vertices.

Therefore, we get $A_{2n+2}(2, 2m, 0, 1) = 2A_{2n+1}(2, 2m, 0, 1)$.

3. The general case

3.1. Generalized Fibonacci and Lucas polynomials.

The Fibonacci polynomials $F_n^{(k)}(x, s)$ are defined by $F_n^{(k)}(x, s) = xF_{n-1}^{(k)}(x, s) + sF_{n-k}^{(k)}(x, s)$ with initial values $F_0^{(k)}(x, s) = 0$ and $F_n^{(k)}(x, s) = x^{n-1}$ for $0 < n < k$.

The Lucas polynomials $L_n^{(k)}(x, s)$ are defined by $L_n^{(k)}(x, s) = xL_{n-1}^{(k)}(x, s) + sL_{n-k}^{(k)}(x, s)$ with initial values $L_0^{(k)}(x, s) = k$ and $L_n^{(k)}(x, s) = x^n$ for $0 < n < k$.

From $(1 - xz - sz^k) \sum_{n \geq 0} F_n^{(k)}(x, s) z^n = (1 - xz - sz^k) (z + xz^2 + \dots + x^{k-1} z^k + \dots) = z$ we get

$$\sum_{n \geq 0} F_{n+1}^{(k)}(x, s) z^n = \frac{1}{1 - xz - sz^k} \quad (14)$$

$$\frac{1}{1 - xz - sz^k} = \sum_{\ell \geq 0} (xz + sz^k)^\ell = \sum_{j, \ell} \binom{\ell}{j} s^j z^{jk} x^{\ell-j} z^{\ell-j} = \sum_{j(k-1) + \ell = n} z^n \sum_{j=0}^{\ell} \binom{\ell}{j} s^j x^{\ell-j} = \sum_{n \geq 0} \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \binom{n - (k-1)j}{j} s^j x^{n-kj}$$

gives

$$F_{n+1}^{(k)}(x, s) = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \binom{n - (k-1)j}{j} s^j x^{n-kj}. \quad (15)$$

From

$$(1 - xz - sz^k) \sum_{n \geq 0} L_n^{(k)}(x, s) z^n = (1 - xz - sz^k) (k + xz + x^2 z^2 + \dots + x^{k-1} z^{k-1} + x^k z^k + \dots) = k - (k-1)xz$$

we get

$$\sum_{n \geq 0} L_n^{(k)}(x, s) z^n = \frac{k - (k-1)xz}{1 - xz - sz^k}. \quad (16)$$

Thus we get $L_n^{(k)}(x, s) = kF_{n+1}^{(k)}(x, s) - (k-1)xF_n^{(k)}(x, s)$.

Since

$$\begin{aligned} k \binom{n - (k-1)j}{j} - (k-1) \binom{n-1 - (k-1)j}{j} &= \binom{n - (k-1)j}{j} + (k-1) \binom{n-1 - (k-1)j}{j-1} \\ &= \binom{n - (k-1)j}{j} \left(1 + (k-1) \frac{j}{n - (k-1)j} \right) = \binom{n - (k-1)j}{j} \frac{n}{n - (k-1)j} \end{aligned}$$

we get for $n \geq k$

$$L_n^{(k)}(x, s) = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \binom{n - (k-1)j}{j} \frac{n}{n - (k-1)j} s^j x^{n-kj}. \quad (17)$$

Other generalizations of $F_n(x, s)$ are the polynomials $G_n^{(k)}(x, s)$ with generating function

$$\sum_{n \geq 0} G_n^{(k)}(x, s) z^n = \frac{1}{1 - s^{k-2} x z^{k-1} - s^{k-1} z^k}. \quad (18)$$

They satisfy $G_n^{(k)}(x, s) = s^{k-2} x G_{n-1}^{(k)}(x, s) + s^{k-1} G_{n-k}^{(k)}(x, s)$ with initial values $G_0^{(k)}(x, s) = 1$, $G_n^{(k)}(x, s) = 0$ for $0 < n \leq k-2$ and $G_{k-1}^{(k)}(x, s) = s^{k-2} x$ and

$$G_n^{(k)}(x, s) = \sum_{j=\lfloor \frac{(k-2)n}{k} \rfloor}^{\frac{(k-1)n}{k}} \binom{n-j}{(k-1)n-kj} s^j x^{(k-1)n-kj}. \quad (19)$$

Proof

$$\frac{1}{1 - s^{k-2} x z^{k-1} - s^{k-1} z^k} = \sum_{\ell \geq 0} (s^{k-2} z^{k-1})^\ell (x + sz)^\ell = \sum_{\ell \geq 0, i \geq 0} \binom{\ell}{i} s^{(k-2)\ell+i} x^{\ell-i} z^{(k-1)\ell+i}$$

If we set $(k-2)\ell + i = j$ and $(k-1)\ell + i = n$ then $\ell = n - j$ and $(k-2)(n-j) + i = j$, which gives

$$n - j - i = n - j - (j - (k-2)(n-j)) = n - 2j + (k-2)n - (k-2)j = (k-1)n - kj.$$

This gives

$$\sum_{\ell \geq 0, i \geq 0} \binom{\ell}{\ell-i} s^{(k-2)\ell+i} x^{\ell-i} z^{(k-1)\ell+i} = \sum_{n \geq 0} z^n \sum_j \binom{n-j}{(k-1)n-kj} s^j x^{(k-1)n-kj},$$

which is (19).

Since $\binom{n}{k} = 0$ for $k < 0$ we can also write

$$\begin{aligned} G_n^{(k)}(x, s) &= \sum_{j=0}^n \binom{n-j}{(k-1)n-kj} s^j x^{(k-1)n-kj} = \sum_{j=0}^n \binom{n-j}{(k-1)j - (k-2)n} s^j x^{(k-1)n-kj} \\ &= \sum_{j=0}^n \binom{n-j}{(k-2)(j-n) + j} s^j x^{(k-1)n-kj} = \sum_{j=0}^n \binom{j}{n - (k-1)j} s^{n-j} x^{kj-n}. \end{aligned} \quad (20)$$

The analog of the Lucas polynomials are the polynomials

$$H_n^{(k)}(x, s) = \sum_{j=\lfloor \frac{(k-2)n}{k} \rfloor}^{\frac{(k-1)n}{k}} \binom{n-j}{(k-1)n-kj} \frac{n}{n-j} s^j x^{(k-1)n-kj}. \quad (21)$$

They can also be written as

$$H_n^{(k)}(x, s) = \sum_{j=0}^{n-1} \binom{n-j}{(k-1)n-kj} \frac{n}{n-j} s^j x^{(k-1)n-kj} = \sum_{j=1}^n \binom{j}{n - (k-1)j} \frac{n}{j} s^{n-j} x^{kj-n}.$$

The initial values are $H_0^{(k)}(x, s) = k$, $H_n^{(k)}(x, s) = 0$ for $0 < n < k-1$, $H_{k-1}^{(k)}(x, s) = (k-1)s^{k-2}x$.

$$(1 - s^{k-2} x z^{k-1} - s^{k-1} z^k) \sum_{n \geq 0} H_n^{(k)}(x, s) z^n = (1 - s^{k-2} x z^{k-1} - s^{k-1} z^k) (k + (k-1)s^{k-2} x z^{k-1} + \dots) = k - s^{k-2} x z^{k-1}$$

implies

$$\sum_{n \geq 0} H_n^{(k)}(x, s) z^n = \frac{k - s^{k-2} x z^{k-1}}{1 - s^{k-2} x z^{k-1} - s^{k-1} z^k}. \quad (22)$$

Remark

The numbers $G_n^{(2)}(1,1)$ are the Fibonacci numbers F_{n+1} . By (18) the numbers

$(G_n^{(3)}(1,1))_{n \geq 0} = (1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, \dots)$ are the Padovan numbers OEIS [11], A000931.

For $4 \leq k \leq 7$ the numbers are listed in OEIS, A017817, A017827, A017837, A017847.

The numbers $H_n^{(2)}(1,1)$ are the Lucas numbers L_n . By (22) the numbers

$(H_n^{(3)}(1,1))_{n \geq 0} = (3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, \dots)$ are the Perrin numbers A001608.

The numbers for $4 \leq k \leq 7$ occur in A050443, A087937, A087936, A306755.

3.2 Recurrences for the general case

The method for $k = 2$ can be generalized to give

Lemma 2

Let

$$\sum_j c(m, k, j) x^j (1+x)^{i_j m - k j} = 0, \tag{23}$$

where i_j is an integer such that $i_j m - k j \geq 0$. Then

$$\sum_j c(m, k, j) z^{i_j} A_{n+i_j m - k j}(k, m, \ell, z) = 0. \tag{24}$$

Proof

Applying the operator $\sum_j c(m, k, j) \Delta^j E^{i_j m - k j}$ to $\left[\begin{matrix} x \\ n + m h + \ell \\ k \end{matrix} \right]$ we get

$$\sum_j c(m, k, j) \Delta^j E^{i_j m - k j} \left[\begin{matrix} x \\ n + m h + \ell \\ k \end{matrix} \right] = \sum_j c(m, k, j) \left[\begin{matrix} x + i_j m - k j \\ n + m h + \ell - k j \\ k \end{matrix} \right] = 0.$$

If we set $x = n$ this gives

$$\sum_j c(m, k, j) \left[\begin{matrix} n + i_j m - k j \\ n + i_j m - k j + m(h - i_j) + \ell \\ k \end{matrix} \right] = 0.$$

Multiplying with z^h and summing over $h \in \mathbb{Z}$ gives

$$\sum_j c(m, k, j) \sum_{h \in \mathbb{Z}} z^h \left[\begin{matrix} n + i(j) m - k j \\ n + i_j m - k j + m(h - i_j) + \ell \\ k \end{matrix} \right] = \sum_j c(m, k, j) z^{i_j} A_{n+i_j m - k j}(k, m, \ell, z) = 0.$$

Let us first consider the case $k = 1$.

Since $-1 + \sum_{j=0}^m \binom{m}{j} (-1)^j x^j (1+x)^{m-j} = 0$ is of the form (23) we get

$$z \sum_{j=0}^m (-1)^j \binom{m}{j} A_{n+m-j}(1, m, \ell, z) = A_n(1, m, \ell, z). \tag{25}$$

For $k > 1$ the following identity of the form (23) has been found in [4]:

There exist uniquely determined integers $a_{m,k,j}$ such that

$$(1+x)^m - 1 + \sum_{i=1}^{k-1} \sum_{j=\left\lceil \frac{(i-1)m}{k-1} \right\rceil}^{\left\lfloor \frac{im}{k} \right\rfloor} a_{m,k,j} x^j (x+1)^{im-kj} = 0. \quad (26)$$

Note that 1 , x^m , and $(1+x)^m$ are of the form $x^j (x+1)^{im-kj}$.

Suppose first that such a formula exists. Then the polynomial $\sum_{j=1}^m a_{m,k,j} z^j$ has the root

$\omega_m^{-k} (\omega_m - 1)$, where ω_m denotes a primitive m -th root of unity. The most obvious polynomial with this root is

$$\prod_{i=1}^m \left(z - \omega_m^{-kj} (\omega_m^j - 1) \right) = \sum_{j=1}^m b_{m,k,j} z^j. \quad (27)$$

This led in [4], Lemma 7.1 to the formula

$$(-1)^{k(m-1)} \sum_{j=1}^m b_{m,k,j} x^j (1+x)^{\text{mod}(-kj,m)} = (x+1)^m - 1, \quad (28)$$

where $\text{mod}(\ell, m)$ denotes the least non-negative residue of ℓ modulo m .

For example for $(k, m) = (3, 4)$ we get $\sum_{j=1}^4 b_{4,3,j} z^j = -4z - 2z^2 + z^4$. This gives

$$4x(1+x) + 2x^2(1+x)^2 - x^4 = 4x(1+x)^{4-3} + 2x^2(1+x)^{8-2\cdot 3} - x^4(1+x)^{12-4\cdot 3} = (x+1)^4 - 1.$$

From another point of view John P. D'Angelo [2] showed that there is a uniquely determined polynomial

$$f_{m,k}(x, y) = 1 - \prod_{j=0}^{m-1} \left(1 - \omega_m^j x - \omega_m^{kj} y \right) \quad (29)$$

which satisfies the following 4 conditions:

- 1) $f(0, 0) = 0$,
- 2) $f(x, y) = 1$ when $x + y = 1$,
- 3) $\deg f = m$,
- 4) $f(\omega_m x, \omega_m^k y) = f(x, y)$,

It turns out that $r(m, k, x, y) = x^m + (-1)^{k(m-1)} \sum_{j=1}^m b_{m,k,j} (-y)^j x^{\text{mod}(-kj,m)}$ satisfies these conditions:

1), 3), 4) are obvious and 2) follows from (28).

This observation led to a simpler approach of (26) in [8], which I will now present in a slightly different form which emphasizes the analogy with the case $k = 2$.

Let $\alpha_1(x, s), \alpha_2(x, s), \dots, \alpha_k(x, s)$ be the roots of the polynomial $a_{k,1}(z, x, s) = z^k - xz^{k-1} - s$ and let

$$a_{k,m}(z, x, s) = \prod_{j=1}^k \left(z - \alpha_j^m(x, s) \right) = \sum_{i=0}^k (-1)^i e_{i,k,m}(x, s) z^{k-i}. \quad (30)$$

To compute the elementary symmetric polynomials $e_{i,k,m}(x, s)$ observe that

$$\prod_{i=1}^k (1 - \alpha_i^m z^m) = \prod_{i=1}^k \prod_{j=0}^{m-1} (1 - \omega_m^j \alpha_i z) = \prod_{j=0}^{m-1} \prod_{i=1}^k (1 - \omega_m^j \alpha_i z) = \prod_{j=0}^{m-1} (1 - \omega_m^j xz - \omega_m^{kj} s z^k) = \Phi_{k,m}(z, x, s).$$

Therefore

$$\prod_{j=0}^{m-1} (1 - \omega_m^j xz - \omega_m^{kj} sz^k) = \prod_{i=1}^k (1 - \alpha_i^m z^m) = \sum_{i=0}^k (-1)^i e_{i,k,m}(x,s) z^{im}. \quad (31)$$

Here $e_{0,k,m}(x,s) = 1$ and

$$e_{k,k,m}(x,s) = (-1)^{(k-1)m} s^m, \quad (32)$$

because $(-1)^k \alpha_1 \alpha_2 \cdots \alpha_k = -s$ and therefore $e_{k,k,m}(x,s) = (\alpha_1 \alpha_2 \cdots \alpha_k)^m = (-1)^{(k-1)m} s^m$. For $m = 0$ we get from (30)

$$e_{i,k,0} = \binom{k}{i}. \quad (33)$$

The left-hand side of (31) is a linear combination of $s^j z^{kj} x^{m-j} z^{m-j} = s^j x^{m-j} z^{m+(k-1)j}$. Only such terms can occur where $m + (k-1)j = im$ for some i . The polynomials $e_{i,k,m}(x,s)$ are linear combinations of $s^j x^{m-j} = s^j x^{im-kj}$. We have $0 \leq j \leq m = in - kj + j$ and $im - kj \geq 0$. This gives $j \leq \frac{im}{k}$ and $j = kj - (i-1)m \geq 0$, i.e. $j \geq \frac{(i-1)m}{k}$.

Therefore for $1 \leq i < k$

$$p_i(m,k,x,s) = (-1)^{i+1} e_{i,k,m}(x,s) = \sum_{j=\lceil \frac{(i-1)m}{k} \rceil}^{\lfloor \frac{im}{k} \rfloor} a(m,k,j) s^j x^{im-kj} \quad (34)$$

for some integers $a(m,k,j)$. The left-hand side of (31) vanishes for $(x+s, z) = (1, 1)$, because the factor for $j = 0$ vanishes. This implies

$$\Phi_{k,m}(1, x+1, -x) = \sum_{i=0}^k (-1)^i e_{i,k,m}(x+1, -x) = 0, \quad (35)$$

or equivalently

$$\sum_{i=1}^{k-1} p_i(m,k,x+1,-x) = 1 + (-1)^{k(m-1)} x^m. \quad (36)$$

By Lemma 2 we get

$$\sum_{i=1}^{k-1} z^i p_i(m,k,N,-1) A_n(k,m,\ell,z) = (1 + (-1)^{k(m-1)} z^k) A_n(k,m,\ell,z). \quad (37)$$

Theorem 3

The characteristic polynomial of the sequence $(A_n(k,m,\ell,z))_{n \geq 0}$ is

$$p_{m,k,z}(t) = (1 + (-1)^{k(m-1)} z^k) - \sum_{i=1}^{k-1} z^i p_i(m,k,t,-1). \quad (38)$$

As shown by Dusty Grundmeier [10] the characteristic polynomials $P_{i,k}(t, x, s)$ of the

sequences $(p_i(m,k,x,s))_{m \geq 0}$ have degree $\binom{k}{i}$.

For $i = 1$ and $i = k-1$ they are explicitly given by

$$\begin{aligned} P_{1,k}(t, x, s) &= t^k - xt^{k-1} - s, \\ P_{k-1,k}(t, x, s) &= t^k + (-1)^{k-1} s^{k-2} xt - s^{k-1}. \end{aligned} \quad (39)$$

In order to verify (39) let us recall Newton's identities

$$p_m - p_{m-1}e_1 + p_{m-2}e_2 - \cdots + (-1)^{m-1}p_1e_{m-1} + (-1)^m m s_m$$

for the power sums $p_m(x_1, \dots, x_k) = \sum_i x_i^m$ in terms of the elementary symmetric polynomials $e_i(x_1, \dots, x_k)$.

Since $p_1(m, k, x, s) = e_{1,k,m}(x, s) = \sum_{i=1}^k \alpha_i(x, s)^m$ and $e_{i,k,1}(x, s)$ is the i -th elementary symmetric polynomial of $\alpha_1, \dots, \alpha_k$, we get $e_{1,k,m}(x, s) = x e_{1,k,m-1}(x, s) + s e_{1,k,m-k}(x, s)$.

From $\prod_{j=0}^{m-1} (1 - \omega_m^j x z) = 1 - x^m z^m$ we get from (31) that for $0 < m < k$ the initial values of

$$e_{1,k,m}(x, s) \text{ are } x^m.$$

This gives for $m > 0$

$$p_1(m, k, x, s) = L_m^{(k)}(x, s). \quad (40)$$

Note that $e_{k-1,k,m}(x, s) = (\alpha_1 \alpha_2 \cdots \alpha_k)^m \left(\frac{1}{\alpha_1^m} + \cdots + \frac{1}{\alpha_k^m} \right)$ and let

$$u(z) = \prod_{i=1}^k (\alpha_1 \alpha_2 \cdots \alpha_k z - \alpha_i) = \prod_{i=1}^k ((-1)^{k-1} s z - \alpha_i) = s^k z^k - (-1)^{k-1} x s^{k-1} z^{k-1} - s.$$

Then $-\frac{1}{s} z^k u\left(\frac{1}{z}\right) = z^k + (-1)^{k-1} s^{k-2} x z - s^{k-1}$ has the roots $\frac{\alpha_1 \alpha_2 \cdots \alpha_k}{\alpha_i}$.

Therefore $e_{k-1,k,m}(x, s)$ has the characteristic polynomial $p(t) = t^k + (-1)^{k-1} s^{k-2} x t - s^{k-1}$

This gives with Newton's identities (cf. [4] and [10])

$$\sum_{m \geq 0} p_{k-1}(m, k, x, s) z^m = \frac{k - (-s)^{k-2} x z^{k-1}}{1 - (-s)^{k-2} x z^{k-1} - s^{k-1} z^k} \quad (41)$$

Therefore, we have

$$p_{k-1}(m, k, x, s) = H_m^{(k)}(x, (-1)^k s). \quad (42)$$

In the general case no explicit formulas for $p_i(m, k, x, s)$ with $2 \leq i < k-1$ are known.

Consider for example $p_2(m, 4, x, s)$. Its characteristic polynomial is $t^6 + s t^4 + s x^2 t^3 - s^2 t^2 - s^3$.

The first terms of $(p_2(m, 4, x, s))_{m \geq 0}$ are

$$(-6, 0, 2s, 3sx^2, -6s^2, -5s^2x^2, 2s^3 - 3s^2x^4, 14s^3x^2, -6s^4 + 8s^3x^4, -18s^4x^2 + 3s^3x^6, 2s^5 - 25s^4x^4, \dots).$$

By (34) we get for $i \in \{0, 1\}$

$$\begin{aligned} p_2(4m+2i, 4, x, s) &= \sum_{j=0}^{m+i} a(4m+2i, 4, 2m+i-j) s^{2m+i-j} x^{4j}, \\ p_2(4m+1+2i, 4, x, s) &= \sum_{j=0}^{m+i} a(4m+1+2i, 4, 2m+i-j) s^{2m+i-j} x^{4j+2}. \end{aligned} \quad (43)$$

Computations suggest the following values for the first coefficients:

$$a(4m, 4, 2m) = -6, \quad a(4m, 4, 2m-1) = \frac{16m}{m-2} \binom{m+1}{4},$$

$$a(4m, 4, 2m-2) = -\frac{16m(4m^2-15)}{(m+3)(m-3)(m-4)} \binom{m+3}{8},$$

$$a(4m, 4, 2m-3) = \frac{32m(8m^2-35)}{(m-5)(m-6)(m-7)} \binom{m+4}{12},$$

$$a(4m+2, 4, 2m+1) = 2, \quad a(4m+2, 4, 2m+1-1) = -\frac{(4m+2)^2}{(m-1)(m-2)} \binom{m+1}{4},$$

$$a(4m+2, 4, 2m+1-2) = \frac{4(4m+2)^2}{(m-3)(m-4)} \binom{m+3}{8},$$

$$a(4m+2, 4, 2m+1-3) = \frac{2(4m+2)^2(-105+8m+8m^2)}{(m-4)(m-5)(m-6)(m-7)} \binom{m+4}{12},$$

$$a(4m+1, 4, 2m) = -m(4m+1), \quad a(4m+1, 4, 2m-1) = \frac{2(4m+1)(4m-3)}{(m-2)(m-3)} \binom{m+2}{6},$$

$$a(4m+1, 4, 2m-2) = -\frac{8(4m+1)(4m^2+9m-10)}{(m-4)(m-5)(m-6)} \binom{m+3}{10}$$

$$a(4m+3, 4, 2m+1) = (m+1)(4m+3), \quad a(4m+3, 4, 2m+1-1) = -\frac{2(4m+3)(4m+7)}{(m-2)(m-3)} \binom{m+2}{6},$$

$$a(4m+3, 4, 2m+1-2) = \frac{8(m+4)(4m+3)(4m^2-m-15)}{(m-3)(m-4)(m-5)(m-6)} \binom{m+3}{10}.$$

Perhaps someone can find a general formula.

Some examples

For $k=3, z=-1$ the characteristic polynomial is

$$p_{m,3,-1}(t) = \left(1 + (-1)^m\right) + p_1(m, 3, t, -1) - p_2(m, 3, t, -1). \quad (44)$$

Here we get $(A_n(3, 1, 0, -1))_{n \geq 0} = (0, 0, 0, \dots)$ with $p_{1,3,-1}(t) = t$,

$(A_n(3, 2, 0, -1))_{n \geq 0} = (1, -2, 2, 0, -4, 8, -8, 0, 16, -32, 32, 0, -64, 128, -128, 0, 256, -512, \dots)$ with

$$p_{2,3,-1}(t) = 2 + 2t + t^2,$$

$(A_n(3, 3, 0, -1))_{n \geq 0} = (0, 0, 0, \dots)$ with $p_{1,3,-1}(t) = t^3$,

$(A_n(3, 4, 0, -1))_{n \geq 0} = (0, 0, 1, -2, -2, 8, -6, -20, 48, 0, -164, 232, 232, -1120, 792, 2704, -6528, 0, \dots)$

with $p_{1,4,-1}(t) = 2 - 4t + 2t^2 + t^4$,

$(A_n(3, 5, 0, -1))_{n \geq 0} = (1, 0, 0, 1, -5, 0, 5, -30, 25, 25, -175, 275, 0, -1000, 2250, -1375, -5000, \dots)$

with $p_{1,5,-1}(t) = 5t - 5t^2 + t^5$.

4. Simpler recurrences for $k=2$ and $z=\pm 1$.

4.1. By (13) the sequence $(A_n(2, m, \ell, -1))_{n \geq 0}$ has characteristic polynomial $L_m(t, -1) + 2$.

The polynomials $L_n(x, -1) + 2$ have non-trivial factors.

$$L_{2n}(x, -1) + 2 = L_n(x, -1)^2, \quad (45)$$

$$L_{2n+1}(x, -1) + 2 = (x + 2)(F_{n+1}(x, -1) - F_n(x, -1))^2. \quad (46)$$

To prove these identities observe that $\alpha(x, -1)\beta(x, -1) = 1$, $(\alpha(x, -1) - \beta(x, -1))^2 = x^2 - 4$,
 $(\alpha(x, -1) - 1)^2 = (x - 2)\alpha(x, -1)$, $(\beta(x, -1) - 1)^2 = (x - 2)\beta(x, -1)$,
 $(\alpha(x, -1) - 1)(\beta(x, -1) - 1) = 2 - x$.

(45) follows from

$$L_m(x, -1)^2 = (\alpha(x, -1)^m + \beta(x, -1)^m)^2 = \alpha(x, -1)^{2m} + 2(\alpha(x, -1)\beta(x, -1))^m + \beta(x, -1)^{2m} = L_{2m}(x, -1) + 2.$$

To verify (46) observe that

$$\begin{aligned} (\alpha - \beta)^2 (F_{m+1}(x, -1) - F_m(x, -1))^2 &= (\alpha^{m+1} - \beta^{m+1} - \alpha^m + \beta^m)^2 = (\alpha^m(\alpha - 1) - \beta^m(\beta - 1))^2 \\ &= \alpha^{2m}(\alpha - 1)^2 + \beta^{2m}(\beta - 1)^2 - 2\alpha^m\beta^m(\alpha - 1)(\beta - 1) = \alpha^{2m}(x - 2)\alpha + \beta^{2m}(x - 2)\beta - 2(2 - x) \\ &= (x - 2)(\alpha^{2m+1} + \beta^{2m+1} + 2). \end{aligned}$$

4.2. Identities (45) and (46) suggest

Theorem 4

$$L_m(N, -1)A_n(2, 2m, \ell, -1) = 0 \quad (47)$$

and

$$(F_{m+1}(N, -1) - F_m(N, -1))A_n(2, 2m + 1, \ell, -1) = 0 \quad (48)$$

for each $\ell \in \mathbb{Z}$.

In these cases Lemma 2 is not applicable because the operators depend on m and not on $2m$ or $2m + 1$. Let us first consider (47). Here the method of Lemma 2 works.

Applying the identity $L_m(E, -\Delta) = \Delta^m + I$ to $\begin{pmatrix} x \\ r \end{pmatrix}$ gives

$$\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^j \binom{m-j}{j} \frac{m}{m-j} \binom{x+m-2j}{r-j} = \binom{x}{r} + \binom{x}{r-m}.$$

Setting $x = n$ and $r = \lfloor \frac{n + \ell - 2mh + m}{2} \rfloor$, multiplying the identity by $(-1)^h$ and summing

over $h \in \mathbb{Z}$ we get

$$\begin{aligned} &\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^j \binom{m-j}{j} \frac{m}{m-j} \sum_{h \in \mathbb{Z}} (-1)^h \binom{n+m-2j}{\lfloor \frac{n + \ell - 2mh - 2j + m}{2} \rfloor} \\ &= \sum_{h \in \mathbb{Z}} (-1)^h \binom{n}{\lfloor \frac{n + \ell - 2mh + m}{2} \rfloor} + \sum_{h \in \mathbb{Z}} (-1)^h \binom{n}{\lfloor \frac{n + \ell - 2m(h+1) + m}{2} \rfloor} \end{aligned} \quad (49)$$

Observing that $\sum_{h \in \mathbb{Z}} (-1)^h \binom{n+m-2j}{\lfloor \frac{n + \ell - 2mh - 2j + m}{2} \rfloor} = A_{n+m-2j}(2, 2m, \ell, -1)$

and that the right-hand side of (49) vanishes we get (47).

For the proof of (48) we recall the proof of [5] which uses an idea of E. Brietzke [3]. Consider the numbers

$$t(n, k) = (-1)^k \binom{n}{\lfloor \frac{n+k}{2} \rfloor}. \quad (50)$$

They satisfy $t(n, k) = -t(n-1, k-1) - t(n-1, k+1)$ with $t(0, 0) = 1, t(0, 1) = -1$ and $t(0, k) = 0$ for all other $k \in \mathbb{Z}$.

Let $s(n, k)$ on $\mathbb{N} \times \mathbb{Z}$ be the function which satisfies the same recurrence, but with initial values $s(0, k) = [k = 0]$. Writing in row n the values $f(n, k)$, we get a signed Pascal's triangle:

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -3 & 0 & -3 & 0 & -1 & 0 \\ 1 & 0 & 4 & 0 & 6 & 0 & 4 & 0 & 1 \end{array}$$

Then the function $t(n, k)$ looks like

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & -2 & 1 & -1 & 0 \\ 0 & -1 & 1 & -3 & 3 & -3 & 3 & -1 & 1 \\ 1 & -1 & 4 & -4 & 6 & -6 & 4 & -4 & 1 \end{array}$$

Define linear operators N and K for functions on $\mathbb{N} \times \mathbb{Z}$ by

$$\begin{aligned} Nf(n, k) &= f(n+1, k), \\ Kf(n, k) &= f(n, k-1). \end{aligned} \quad (51)$$

They give

$$s(n, k) = Ns(n-1, k) = -s(n-1, k-1) - s(n-1, k+1) = -(K + K^{-1})s(n-1, k) = (-1)^n (K + K^{-1})^n s(0, k).$$

The initial values give $t(0, k) = (1 - K)s(0, k)$ and linearity gives $t(n, k) = (1 - K)s(n, k)$.

Let \mathcal{F} be the linear space of all functions on $\mathbb{N} \times \mathbb{Z}$ which are finite linear combinations of functions $K^j s(n, k)$ for $j \in \mathbb{Z}$.

For $f \in \mathcal{F}$ we have

$$Nf = -(K + K^{-1})f. \quad (52)$$

For $x \rightarrow -x, y \rightarrow -\frac{1}{x}$ in (10) we get $F_n\left(-x - \frac{1}{x}, -1\right) = (-1)^{n-1} \sum_{j=0}^{n-1} x^{n-1-2j}$ and

$$F_{n+1}\left(-x - \frac{1}{x}, -1\right) - F_n\left(-x - \frac{1}{x}, -1\right) = (-1)^n \sum_{j=-n}^n x^j.$$

Therefore on \mathcal{F} we have

$$(F_{m+1}(N, -1) - F_m(N, -1)) = (-1)^m \sum_{j=-m}^m K^j. \quad (53)$$

This implies $(F_{m+1}(N, -1) - F_m(N, -1))s(0, k) = (-1)^m [k \leq m]$ and

$$(F_{m+1}(N, -1) - F_m(N, -1)) \sum_{h \in \mathbb{Z}} s(n, \ell - (2m+1)h) = (-1)^m.$$

Since $t(n, k) = (1 - K)s(n, k)$ we also have

$$(F_{m+1}(N, -1) - F_m(N, -1)) \sum_{h \in \mathbb{Z}} t(n, \ell - (2m+1)h) = 0. \quad (48) \text{ follows from}$$

$$\sum_{h \in \mathbb{Z}} t(n, \ell - (2m+1)h) = (-1)^\ell \sum_{h \in \mathbb{Z}} (-1)^h \left(\left[\frac{n}{\frac{n + \ell - (2m+1)h}{2}} \right] \right) = (-1)^\ell A_n(2, 2m+1, \ell, -1).$$

4.3. For $z = 1$ there are also simpler recurrences.

Theorem 5

$$(N - 2)F_m(N, -1)A_n(2, 2m, \ell, 1) = 0 \quad (54)$$

and

$$(L_{m+1}(N, -1) - L_m(N, -1))A_n(2, 2m+1, \ell, 1) = 0. \quad (55)$$

Remark

Note that

$$L_{2m}(x, -1) - 2 = (x^2 - 4)F_m(x, -1)^2 \quad (56)$$

and

$$L_{2m+1}(x, -1) - 2 = \frac{(L_{m+1}(x, -1) - L_m(x, -1))^2}{x - 2}. \quad (57)$$

Formula (56) follows from

$$(\alpha(x, -1)^m - \beta(x, -1)^m)^2 = \alpha(x, -1)^{2m} + \beta(x, -1)^{2m} - 2(\alpha(x, -1)\beta(x, -1))^m.$$

Formula (57) follows from

$$\begin{aligned} (L_{m+1}(x, -1) - L_m(x, -1))^2 &= (\alpha^{m+1} + \beta^{m+1} - \alpha^m - \beta^m)^2 = (\alpha^m(\alpha - 1) + \beta^m(\beta - 1))^2 \\ &= \alpha^{2m}(\alpha - 1)^2 + \beta^{2m}(\beta - 1)^2 + 2\alpha^m\beta^m(\alpha - 1)(\beta - 1) = \alpha^{2m}(x - 2)\alpha + \beta^{2m}(x - 2)\beta + 2(2 - x) \\ &= (x - 2)(\alpha^{2m+1} + \beta^{2m+1} - 2) \end{aligned}$$

Proof of Theorem 5.

The function $u(n, \ell) = \left[\left[\frac{n}{\frac{n + \ell}{2}} \right] \right]$ satisfies $u(n, \ell) = u(n - 1, \ell - 1) + u(n - 1, \ell + 1)$ and

$u(0, \ell) = [\ell \in \{0, 1\}]$. Therefore $u(n, \ell) = (1 + K)S(0, \ell)$, where $S(n, \ell)$ satisfies the same recurrence as $u(n, \ell)$ but with initial values $S(0, \ell) = [\ell = 0]$.

Let \mathcal{G} be the linear space of all finite linear combinations $K^j S(n, \ell)$ for $j \in \mathbb{Z}$. On \mathcal{G} we have

$$N = K + K^{-1}.$$

The identity

$$\left(x + \frac{1}{x} - 2 \right) F_m \left(x + \frac{1}{x}, -1 \right) (1 + x) = \frac{(x - 1)^2}{x} \left(\frac{x^m - \frac{1}{x^m}}{x - \frac{1}{x}} \right) (1 + x) = (x - 1) \left(x^m - \frac{1}{x^m} \right) = \frac{1}{x^m} - \frac{1}{x^{m-1}} - x^m + x^{m+1}$$

implies

$$(E - 2)F_m(E, -1)u(0, \ell) = (K^m - K^{m-1} - K^{-m} + K^{-m-1})S(0, \ell)$$

and therefore

$$(E-2)F_m(E, -1) \sum_{h \in \mathbb{Z}} K^{2hm} u(0, \ell) = \sum_{h \in \mathbb{Z}} K^{2hm} (K^m - K^{m-1} - K^{-m} + K^{-m-1}) S(0, \ell) = 0.$$

This gives (54).
From

$$\left(L_m \left(x + \frac{1}{x}, -1 \right) - L_{m-1} \left(x + \frac{1}{x}, -1 \right) \right) (1+x) = (1+x) \left(x^m + \frac{1}{x^m} - x^{m-1} - \frac{1}{x^{m-1}} \right) = \frac{1}{x^m} - \frac{1}{x^{m-2}} - x^{m-1} + x^{m+1}$$

we get (55).

5. Some final remarks

It is perhaps interesting to note that (30) also gives the recurrences of subsequences of the generalized Fibonacci polynomials $F_n^{(k)}(x, s)$.

By (30) the characteristic polynomials of the subsequences $(F_{mn+r}^{(k)}(x, s))$ are given by

$$q_{m,k}(t) = \sum_{i=0}^k (-1)^i e_{i,k,m}(x, s) t^{k-i}. \quad (58)$$

For $k=3$ this reduces to

$$q_{m,3}(t) = t^3 - e_{1,3,m}(x, s)t^2 + e_{2,3,m}(x, s)t - e_{3,3,m}(x, s). \quad (59)$$

By (40)

$$e_{1,3,m}(x, s) = \sum_{j=0}^{\lfloor \frac{m}{3} \rfloor} \binom{m-2j}{j} \frac{m}{m-2j} s^j x^{m-3j}. \quad (60)$$

The first terms are $3, x, x^2, x^3 + 3s, x^4 + 4sx, x^5 + 5sx^2, x^6 + 6sx^3 + 3s^2, \dots$.

By (42)

$$e_{2,3,m}(x, s) = \sum_{j=\lceil \frac{m}{3} \rceil}^{\lfloor \frac{2m}{3} \rfloor} (-1)^j \binom{m-j}{2m-3j} \frac{m}{m-j} s^j x^{2m-3j}. \quad (61)$$

The first terms are $3, 0, -2sx, 3s^2, 2s^2x^2, -5s^3x, 3s^4 - 2s^3x^3, 7s^4x^2, \dots$.

By (32)

$$e_{3,3,m} = s^m. \quad (62)$$

Remark

The sequence $(G_n)_{n \geq 0} = (1, 1, 1, 2, 3, 4, 6, 9, 13, 19, \dots)$ with $G_n = F_{n+1}^{(3)}(1, 1)$ is called Narayana's cows sequence, OEIS [11], A000930.

From (59) we get the recurrences $G_{m(n+3)} = a_m G_{m(n+2)} - b_m G_{m(n+1)} + G_{mn}$, where $a_m = e_{1,3,m}(1, 1)$ satisfies $a_m = a_{m-1} + a_{m-3}$ with initial values $a_0 = 3, a_1 = 1, a_2 = 1$ and $b_m = e_{2,3,m}(1, 1)$ satisfies $b_m = -b_{m-2} + b_{m-3}$ with initial values $b_0 = 3, b_1 = 0, b_2 = -2$.

The first terms are $(a_m)_{m \geq 1} = (1, 1, 4, 5, 6, 10, 15, 21, \dots)$ and $(b_m)_{m \geq 1} = (0, -2, 3, 2, -5, 1, 7, -6, \dots)$.

For example, we get $G_{2n} = G_{2n-2} + 2G_{2n-4} + G_{2n-6}$ and $G_{3n} = 4G_{3n-3} - 3G_{3n-6} + G_{3n-9}$.

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